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L-fuzzy soft sets based on complete Boolean lattices[☆]

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ABSTRACT

In this paper, the concept of L-fuzzy soft sets based on complete Boolean lattices, which can be seen as a generalization of fuzzy soft sets, is introduced. The topological and lattice structure of L-fuzzy soft sets are obtained. Some operations on L-fuzzy soft sets are discussed, and new types of L-fuzzy soft sets such as full, keeping intersection and keeping union L-fuzzy soft sets are defined and supported by some illustrative examples. A pair of L-fuzzy soft rough approximations is proposed and their properties are given. Based on L-fuzzy soft rough approximations, the concept of L-fuzzy soft rough sets is introduced and their structures are studied. The fact that every finite L-fuzzy topological space is an L-fuzzy soft approximation space is revealed. In addition, we obtain two one-to-one correspondence relationships associated with L-fuzzy soft sets, which expounds the broad application prospect of L-fuzzy soft sets.

1. Introduction

Molodtsov [1] proposed soft set theory as a new mathematical tool for dealing with uncertainties which traditional mathematical tools cannot handle. He has shown several applications of this theory in solving many practical problems in economics, engineering, social science, medical science, etc.

Presently, works on soft set theory are progressing rapidly. Maji et al. [2] further studied the theory of soft sets. Aktas and Cağman [3] introduced the concept of soft groups. Jiang et al. [4] extended soft sets with description logics. Feng et al. [5,6] and Ali [7] investigated the relationship among soft sets, rough sets and fuzzy sets. Ge et al. [8] discussed the relationship between soft sets and topological spaces.

In soft set theory, we observe that in most cases the parameters are vague words or sentences involve vague words. Considering this point, Maji et al. [9] proposed the concept of fuzzy soft sets by combining soft sets with fuzzy sets. Roy and Maji [10]presented a fuzzy soft set theoretical approach towards decision making problems. Feng et al. [11] proposed an adjustable approach to (weighted) fuzzy soft set based decision making. Yang et al. [12] introduced the concept of interval-valued fuzzy soft sets. Tanay and Kandemir [13] investigated the topological structure of fuzzy soft sets.

Algebraic and topological structures play a fundamental role in many fields of mathematics. This paper is devoted to the discussion of topological and algebraic structures of L-fuzzy soft sets. Its remaining parts are organized as follows. In Section 2, we recall several basic concepts of lattices and L-fuzzy sets. In Section 3, we introduce the concept of L-fuzzy soft sets based on complete Boolean lattices as a generalization of fuzzy soft sets, discuss some operations on L-fuzzy soft sets and define new types of L-fuzzy soft sets. In Section 4, we obtain the algebraic structure (i.e., the lattice structure) of L-fuzzy

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soft sets. In Section 5, we consider a pair of L-fuzzy soft rough approximations and give their properties. Based on L-fuzzy soft rough approximations, we introduce the concept of L-fuzzy soft rough sets and obtain the structure of L-fuzzy soft rough sets. In Section 6, we investigate the relationship between L-fuzzy soft sets and L-fuzzy topologies, obtain the topological structure of L-fuzzy soft sets, and reveal the fact that every finite L-fuzzy topological space is an L-fuzzy soft approximating space. In Section 7, we get two one-to-one correspondence relationships associated with L-fuzzy soft sets. The conclusion is in Section 8.

2. Preliminaries

We briefly recall several basic concepts of lattices and L-fuzzy sets.

Definition 2.1 (*[14]*)**.** Let $(L, \leq)$ be a poset.

(1) L is called a lattice, if $a \vee b \in L,\ a \wedge b \in L$ for any $a, b \in L$.
(2) L is called a complete lattice, if $\vee S \in L,\ \ \wedge\ S \in L$ for any $S \subseteq L$.
(3) L is called distributive, if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ for any $a, b, c \in L$.
(4) L is called a complete distributive lattice (resp. a distributive lattice), if L is a complete lattice (resp. a lattice) and distributive.

Obviously, every complete lattice (resp. a complete distributive lattice) is a lattice (resp. a distributive lattice).
In this paper, we stipulate that $\vee\emptyset = 0_L$.

Example 2.2. Let $L = [0, 1]$. For any $a, b \in L$, we defined $a \leq b$ by $b - a \geq 0$.
It is easily proved that L is a complete distributive lattice with $1_L = 1$ and $0_L = 0$.

Definition 2.3 (*[14]*)**.** Let L be a lattice with top element 1_L and bottom element 0_L and let $a, b \in L$. b is called a complement element of a, if $a \vee b = 1_L$, $a \wedge b = 0_L$.

If $a \in L$ has a complement element, then it is unique. We denote the complement element of a by a'.

Definition 2.4 (*[14]*)**.** Let $(L, \leq)$ be a poset.

(1) L is called a Boolean lattice, if (i) L is a distributive lattice; (ii) L has 0_L and 1_L; (iii) each $a \in L$ has the complement $a' \in L$.
(2) L is called a complete Boolean lattice, if (i) L is a complete distributive lattice; (ii) L has 0_L and 1_L; (iii) each $a \in L$ has the complement $a' \in L$.

Obviously, every complete Boolean lattice is a Boolean lattice.
In this paper, top element 1_L and bottom element 0_L are also denoted by 1 and 0, respectively.

Proposition 2.5 (*[14]*)**.** *Let L be a Boolean lattice. Then*

(1) $0' = 1$ *and* $1' = 0$.
(2) $a'' = a$ *for each* $a \in L$.
(3) $(a \vee b)' = a' \wedge b'$ *for any* $a, b \in L$.
(4) $(a \wedge b)' = a' \vee b'$ *for any* $a, b \in L$.
(5) $a \leq b$ *if and only if* $a \wedge b' = 0$ *for any* $a, b \in L$.

In [15], Goguen first introduced a L-fuzzy set as a generalization of Zadeh's fuzzy set. In what follows, we recall the specific definitions of L-fuzzy sets.

Definition 2.6 (*[15]*)**.** Let X be a common set and let L be a complete distributive lattice with 1 and 0. An L-fuzzy set A in X is defined by a map $A : X \to L$.

We denote the family of all L-fuzzy sets in X by L^X. For every $a \in L$, $\widetilde{a}$ denotes the constant L-fuzzy set.
For any $A, B \in L^X$, we define $A \subseteq B$ by $A(x) \leq B(x)$ for every $x \in X$.
From given L-fuzzy sets A and B, new L-fuzzy sets can be constructed as follows:

$$(A \cap B)(x) = A(x) \wedge B(x);$$

$$(A \cup B)(x) = A(x) \vee B(x);$$

$$A^c(x) = A(x)' \quad \text{for each } x \in X \text{ when } L \text{ is a Boolean lattice.}$$

Moreover, $(\bigcup_{\alpha\in\Gamma} A_\alpha)(x) = \bigvee_{\alpha\in\Gamma} A_\alpha(x)$ or $\bigvee\{A_\alpha(x) : \alpha \in \Gamma\}$ for each $x \in X$ and $(\bigcap_{\alpha\in\Gamma} A_\alpha)(x) = \bigwedge_{\alpha\in\Gamma} A_\alpha(x)$ or $\bigwedge\{A_\alpha(x) : \alpha \in \Gamma\}$ for each $x \in X$, where $\{A_\alpha : \alpha \in \Gamma\} \subseteq L^X$.

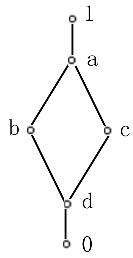


Fig. 1

Obviously,

$$\widetilde{0} \subseteq A \subseteq \widetilde{1} \quad \text{for any } A \in L^X.$$

An L-fuzzy set is called an L-fuzzy singleton in X, if it takes the value 0 for each $y \in X$ except one, say, $x \in X$. If its value at x is $\lambda(0 < \lambda \leq 1)$, we denote this L-fuzzy singleton by x_λ, where the point x is called its support and λ is called its order height (see [16]).

Example 2.7. Let L be a complete distributive lattice with 1 and 0 depicted in Fig. 1 and let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$.

(1) Put $A(x_1) = a, A(x_2) = c, A(x_3) = 1, A(x_4) = a, A(x_5) = d, A(x_6) = 0$. Then A is an L-fuzzy set in X. We denote it by

$$A = \left\{ \frac{x_1}{a}, \frac{x_2}{c}, \frac{x_3}{1}, \frac{x_4}{a}, \frac{x_5}{d}, \frac{x_6}{0} \right\}.$$

(2) Put $B = \left\{ \frac{x_1}{b}, \frac{x_2}{b}, \frac{x_3}{d}, \frac{x_4}{1}, \frac{x_5}{b}, \frac{x_6}{a} \right\}$. Then

$$A \cup B = \left\{ \frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{1}, \frac{x_4}{1}, \frac{x_5}{b}, \frac{x_6}{a} \right\}, \qquad A \cap B = \left\{ \frac{x_1}{b}, \frac{x_2}{d}, \frac{x_3}{d}, \frac{x_4}{a}, \frac{x_5}{d}, \frac{x_6}{0} \right\}.$$

(3) Put $C = \left\{ \frac{x_1}{0}, \frac{x_2}{b}, \frac{x_3}{0}, \frac{x_4}{0}, \frac{x_5}{0}, \frac{x_6}{0} \right\}$. Then

C is an L-fuzzy singleton in X.

Pick $\lambda = b$, then $x_\lambda = C$.

For an L-fuzzy singleton x_λ and $A \in L^X$, we define $x_\lambda \in A$ by $x_\lambda \subseteq A$.
Obviously,

$$x_\lambda \in A \quad \text{if and only if} \quad \lambda \leq A(x).$$

Definition 2.8 (*[17]*)**.** Let L be a complete Boolean lattice and $\tau \subseteq L^X$. If τ satisfies the following conditions:
(i) $\widetilde{1}, \widetilde{0} \in \tau$; (ii) $A, B \in \tau$ implies $A \cap B \in \tau$; and (iii) $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau$ implies $\cup\{A_\alpha : \alpha \in \Gamma\} \in \tau$.
Then τ is called an L-fuzzy topology on X, and the pair (X, τ) is called an L-fuzzy topological space and every member of τ is called an L-fuzzy open set in X. $A \in L^X$ is called an L-fuzzy closed set in X if $A^c \in \tau$.

Let (X, τ) be an L-fuzzy topological space and let $A \in L^X$. Then the interior of A and the closure of A, denoted respectively by $\text{int}(A)$ and $\text{cl}(A)$, are defined as follows:

$$\text{int}(A) = \cup\{B : B \subseteq A \text{ and } B \in \tau\}; \qquad \text{cl}(A) = \cap\{B : B \supseteq A \text{ and } B^c \in \tau\}.$$

Definition 2.9 (*[18,19]*)**.** Let L be a complete Boolean lattice and $\tau \subseteq L^X$. If τ satisfies the following conditions:
(i) $\widetilde{0} \in \tau$; and (ii) $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau$ implies $\cup\{A_\alpha : \alpha \in \Gamma\} \in \tau$.
Then τ is called a generalized L-fuzzy topology on X, and the pair (X, τ) is called a generalized L-fuzzy topological space and every member of τ is called a generalized L-fuzzy open set in X. $A \in L^X$ is called a generalized L-fuzzy closed set in X if $A^c \in \tau$.

3. *L*-fuzzy soft sets

Let L be a complete Boolean lattice. X denotes the universe, E denotes the set of parameters, which are attributes, characteristics or properties of the objects in X. Throughout this paper, we always assume that X and E are both finite sets.

Definition 3.1 (*[9]*)**.** A pair (f, E) is called a fuzzy soft set over X, if f is a mapping given by $f : E \to I^X$. We also denote (f, E) by f_E.

Table 1
Tabular representation of the fuzzy soft set f_E.

	x_1	x_2	x_3	x_4	x_5	x_6
e_1	0	0.3	0.8	0.5	0.7	0.3
e_2	0.7	0.5	0.1	0.2	0.2	0.6
e_3	0.1	0.9	1	0.5	0.1	0.7

Table 2
Tabular representation of the L-fuzzy soft set f_E.

	x_1	x_2	x_3	x_4	x_5	x_6
e_1	0	b	c	d	f	b
e_2	f	d	1	a	a	e
e_3	1	f	1	d	1	f
e_4	d	1	1	c	0	1

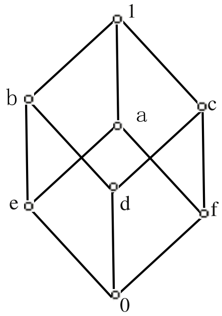


Fig. 2

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $E = \{e_1, e_2, e_3\}$. Let f_E be a fuzzy soft set over X, defined as follows

$$f(e_1) = \left\{\frac{x_1}{0}, \frac{x_2}{0.3}, \frac{x_3}{0.8}, \frac{x_4}{0.5}, \frac{x_5}{0.7}, \frac{x_6}{0.3}\right\}, \qquad f(e_2) = \left\{\frac{x_1}{0.7}, \frac{x_2}{0.5}, \frac{x_3}{0.1}, \frac{x_4}{0.2}, \frac{x_5}{0.2}, \frac{x_6}{0.6}\right\},$$
$$f(e_3) = \left\{\frac{x_1}{0.1}, \frac{x_2}{0.9}, \frac{x_3}{1}, \frac{x_4}{0.5}, \frac{x_5}{0.1}, \frac{x_6}{0.7}\right\}.$$

Then f_E is described as the following Table 1.

Definition 3.3. Let L be a complete Boolean lattice. A pair (f, E) is called an L-fuzzy soft set over X, if f is a mapping given by $f : E \to L^X$. We also denote (f, E) by f_E.

In other words, an L-fuzzy soft set f_E over X is a parameterized family of L-fuzzy sets in the universe X.
If $L = [0, 1]$, then every L-fuzzy soft set is a fuzzy soft set.

Example 3.4. Let L be a complete Boolean lattice depicted in Fig. 2, $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $E = \{e_1, e_2, e_3, e_4\}$. Let f_E be a fuzzy soft set over X. We define as follows:

$$f(e_1) = \left\{\frac{x_1}{0}, \frac{x_2}{b}, \frac{x_3}{c}, \frac{x_4}{d}, \frac{x_5}{f}, \frac{x_6}{b}\right\}, \qquad f(e_2) = \left\{\frac{x_1}{f}, \frac{x_2}{d}, \frac{x_3}{1}, \frac{x_4}{a}, \frac{x_5}{a}, \frac{x_6}{e}\right\},$$
$$f(e_3) = \left\{\frac{x_1}{1}, \frac{x_2}{f}, \frac{x_3}{1}, \frac{x_4}{d}, \frac{x_5}{1}, \frac{x_6}{f}\right\}, \qquad f(e_4) = \left\{\frac{x_1}{d}, \frac{x_2}{1}, \frac{x_3}{1}, \frac{x_4}{c}, \frac{x_5}{0}, \frac{x_6}{1}\right\}.$$

Then f_E is described as the following Table 2.

Definition 3.5. Let $A, B \subseteq E$ and let f_A and g_B be two L-fuzzy soft sets over X.

(1) f_A is called an L-fuzzy soft subset of g_B, if $A \subseteq B$ and $f(e) \subseteq g(e)$ for each $e \in A$. We denote it by $f_A \widetilde{\subseteq} g_B$.
(2) f_A and g_B are called L-fuzzy soft equal, if f_A is an L-fuzzy soft subset of g_B and g_B is an L-fuzzy soft subset of f_A. We denote it by $f_A = g_B$.

Definition 3.6. Let $A \subseteq E$ and let f_A be an L-fuzzy soft set over X.

(1) f_A is called null, if $f(e) = \widetilde{0}$ for any $e \in A$. We denote it by $\emptyset_A$.
(2) f_A is called absolute, if $f(e) = \widetilde{1}$ for any $e \in A$. We denote it by X_A.

We stipulate that $\emptyset_\emptyset$ is also an L-fuzzy soft set over X with $\emptyset : \emptyset \to L^X$.
Let $A \subseteq E$ and let f_A be an L-fuzzy soft set over X. Obviously,

$$\emptyset_A \,\widetilde{\subset}\, f_A \,\widetilde{\subset}\, X_A.$$

Below, we introduce some operations on L-fuzzy soft sets and investigate their properties.

Definition 3.7. Let $A, B \subseteq E$ and let f_A and g_B be two L-fuzzy soft sets over X.

(1) h_C is called the intersection of f_A and g_B, if $C = A \cap B$ and $h(e) = f(e) \cap g(e)$ for each $e \in C$. We denote it by $f_A \,\widetilde{\cap}\, g_B = h_C$.
(2) h_C is called the union of f_A and g_B, if $C = A \cup B$ and

$$h(e) = \begin{cases} f(e), & \text{if } e \in A - B, \\ g(e), & \text{if } e \in B - A, \\ f(e) \cup g(e), & \text{if } e \in A \cap B. \end{cases}$$

We denote it by $f_A \,\widetilde{\cup}\, g_B = h_C$.
(3) h_C is called the bi-intersection of f_A and g_B, if $C = A \times B$ and $h(a, b) = f(a) \cap g(b)$ for each $a \in A$ and $b \in B$. We denote it by $f_A \,\widetilde{\wedge}\, g_B = h_C$.
(4) h_C is called the bi-union of f_A and g_B, if $C = A \times B$ and $h(a, b) = f(a) \cup g(b)$ for each $a \in A$ and $b \in B$. We denote it by $f_A \,\widetilde{\vee}\, g_B = h_C$.

Definition 3.8. Let $A \subseteq E$ and let f_A be an L-fuzzy soft set over X. The complement of f_A is denoted by $(f_A)^c$ and is defined by ${f^c}_A$ or $(f_A)^c = (f^c, A)$, where $f^c : A \to L^X$ is a mapping given by $f^c(e) = f(e)^c$ for each $e \in A$.

Proposition 3.9. *Let $A, B, C \subseteq E$ and let f_A, g_B and h_C be three L-fuzzy soft sets over X. Then*

(1) $f_A \,\widetilde{\cup}\, f_A = f_A$.
(2) $f_A \,\widetilde{\cup}\, g_B = g_B \,\widetilde{\cup}\, f_A$.
(3) $(f_A \,\widetilde{\cup}\, g_B) \,\widetilde{\cup}\, h_C = f_A \,\widetilde{\cup}\, (g_B \,\widetilde{\cup}\, h_C)$.

Proof. (1) and (2) are obvious. We only prove (3). Put

$$(f_A \,\widetilde{\cup}\, g_B) \,\widetilde{\cup}\, h_C = k_{A\cup B\cup C}, \qquad f_A \,\widetilde{\cup}\, (g_B \,\widetilde{\cup}\, h_C) = l_{A\cup B\cup C};$$
$$f_A \,\widetilde{\cup}\, g_B = s_{A\cup B}, \qquad g_B \,\widetilde{\cup}\, h_C = t_{B\cup C}.$$

For any $e \in A \cup B \cup C$, it follows that $e \in A$, or $e \in B$, or $e \in C$.
Case 1 $e \in C$.

(a) If $e \notin A$ and $e \notin B$, then $k(e) = h(e) = t(e) = l(e)$.
(b) If $e \notin A$ and $e \in B$, then $k(e) = s(e) \cup h(e) = g(e) \cup h(e) = t(e) = l(e)$.
(c) If $e \in A$ and $e \notin B$, then $k(e) = s(e) \cup h(e) = f(e) \cup h(e) = f(e) \cup t(e) = l(e)$.
(d) If $e \in A$ and $e \in B$, then $k(e) = s(e) \cup h(e) = f(e) \cup g(e) \cup h(e) = f(e) \cup t(e) = l(e)$.

Case 2 $e \notin C$.

(a) If $e \notin A$ and $e \in B$, then $k(e) = s(e) = g(e) = t(e) = l(e)$.
(b) If $e \in A$ and $e \notin B$, then $k(e) = s(e) = f(e) = l(e)$.
(c) If $e \in A$ and $e \in B$, then $k(e) = s(e) = f(e) \cup g(e) = f(e) \cup t(e) = l(e)$.

Thus $(f_A \,\widetilde{\cup}\, g_B) \,\widetilde{\cup}\, h_C = f_A \,\widetilde{\cup}\, (g_B \,\widetilde{\cup}\, h_C)$. □

Proposition 3.10. *Let $A, B, C \subseteq E$ and let f_A, g_B and h_C be three L-fuzzy soft sets over X. Then*

(1) $f_A \,\widetilde{\cap}\, f_A = f_A$.
(2) $f_A \,\widetilde{\cap}\, g_B = g_B \,\widetilde{\cap}\, f_A$.
(3) $(f_A \,\widetilde{\cap}\, g_B) \,\widetilde{\cap}\, h_C = f_A \,\widetilde{\cap}\, (g_B \,\widetilde{\cap}\, h_C)$.

Proof. (1) and (2) are obvious. We only prove (3). Put

$$(f_A \,\widetilde{\cap}\, g_B) \,\widetilde{\cap}\, h_C = k_{A\cap B\cap C}, \qquad f_A \,\widetilde{\cap}\, (g_B \,\widetilde{\cap}\, h_C) = l_{A\cap B\cap C}.$$

For any $e \in A \cap B \cap C$, it follows that $e \in A$, $e \in B$ and $e \in C$. Since $k(e) = (f(e) \cap g(e)) \cap h(e) = f(e) \cap (g(e) \cap h(e)) = l(e)$, then $(f_A \,\widetilde{\cap}\, g_B) \,\widetilde{\cap}\, h_C = f_A \,\widetilde{\cap}\, (g_B \,\widetilde{\cap}\, h_C)$. □

Proposition 3.11. *Let $A, B, C \subseteq E$ and let f_A, g_B and h_C be three L-fuzzy soft sets over X. Then*

(1) $(f_A \,\widetilde{\vee}\, g_B) \,\widetilde{\vee}\, h_C = f_A \,\widetilde{\vee}\, (g_B \,\widetilde{\vee}\, h_C)$.
(2) $(f_A \,\widetilde{\wedge}\, g_B) \,\widetilde{\wedge}\, h_C = f_A \,\widetilde{\wedge}\, (g_B \,\widetilde{\wedge}\, h_C)$.

Proof. (1) Put

$$(f_A \,\widetilde{\vee}\, g_B) \,\widetilde{\vee}\, h_C = k_{A\times B\times C}, \qquad f_A \,\widetilde{\vee}\, (g_B \,\widetilde{\vee}\, h_C) = l_{A\times B\times C}.$$

For any $(a, b, c) \in A \times B \times C$, it follows that $a \in A$, $b \in B$ and $c \in C$. Since $k(a, b, c) = (f(a) \vee g(b)) \vee h(c) = f(a) \vee (g(b) \vee h(c)) = l(a, b, c)$, then

$$(f_A \,\widetilde{\vee}\, g_B) \,\widetilde{\vee}\, h_C = f_A \,\widetilde{\vee}\, (g_B \,\widetilde{\vee}\, h_C).$$

(2) This is similar to the proof of (1). □

Proposition 3.12. *Let $A, B, C \subseteq E$ and let f_A, g_B and h_C be three L-fuzzy soft sets over X. Then*

(1) $(f_A \,\widetilde{\cup}\, g_B) \,\widetilde{\cap}\, h_C = (f_A \,\widetilde{\cap}\, h_C) \,\widetilde{\cup}\, (g_B \,\widetilde{\cap}\, h_C)$.
(2) $(f_A \,\widetilde{\cap}\, g_B) \,\widetilde{\cup}\, h_C = (f_A \,\widetilde{\cup}\, h_C) \,\widetilde{\cap}\, (g_B \,\widetilde{\cup}\, h_C)$.

Proof. (1) Put

$$(f_A \,\widetilde{\cup}\, g_B) \,\widetilde{\cap}\, h_C = k_{(A\cup B)\cap C}, \qquad (f_A \,\widetilde{\cap}\, h_C) \,\widetilde{\cup}\, (g_B \,\widetilde{\cap}\, h_C) = l_{(A\cap C)\cup(B\cap C)}.$$

Obviously, $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. For any $e \in (A \cup B) \cap C$, it follows that $e \in A \cap C$, or $e \in B \cap C$.
(1) If $e \notin A \cap C$ and $e \in B \cap C$, then $e \notin A$, $e \in B$ and $e \in C$. So $k(e) = g(e) \cap h(e) = l(e)$.
(2) If $e \in A \cap C$ and $e \notin B \cap C$, then $e \in A$, $e \notin B$ and $e \in C$. So $k(e) = f(e) \cap h(e) = l(e)$.
(3) If $e \in A\cap C$ and $e \in B\cap C$, then $e \in A$, $e \in B$ and $e \in C$. So $k(e) = (f(e)\cup g(e))\cap h(e) = (f(e)\cap h(e))\cup(g(e)\cap h(e)) = l(e)$.
Thus

$$(f_A \,\widetilde{\cup}\, g_B) \,\widetilde{\cap}\, h_C = (f_A \,\widetilde{\cap}\, g_B) \,\widetilde{\cup}\, (g_B \,\widetilde{\cap}\, h_C).$$

(2) This is similar to the proof of (1). □

Proposition 3.13. *Let $A, B \subseteq E$ and let f_A and g_B be two L-fuzzy soft sets over X. Then*

(1) $((f_A)^c)^c = f_A$.
(2) $f_A \,\widetilde{\cup}\, (f_A)^c = X_A$.
(3) $f_A \,\widetilde{\cap}\, (f_A)^c = \emptyset_A$.
(4) $(f_A \,\widetilde{\cup}\, g_A)^c = (f_A)^c \,\widetilde{\cap}\, (g_A)^c$.
(5) $(f_A \,\widetilde{\cap}\, g_A)^c = (f_A)^c \,\widetilde{\cup}\, (g_A)^c$.

Proof. (1) Put $(f_A)^c = g_A$, $(g_A)^c = h_A$.
For any $e \in A$, $h(e) = g(e)^c$, $g(e) = f(e)^c$. And for any $x \in X$, $h(e)(x) = (g(e)^c)(x) = (g(e)(x))'$, $g(e)(x) = f(e)^c(x) = (f(e)(x))'$. Then $h(e)(x) = (f(e)(x))''$. By Proposition 2.5, $h(e)(x) = f(e)(x)$. Thus $h(e) = f(e)$. This show that $h_A = f_A$. That is, $((f_A)^c)^c = f_A$.
(2) Put

$$f_A \,\widetilde{\cup}\, (f_A)^c = h_A.$$

For any $e \in A$, $h(e) = f(e) \cup f^c(e) = f(e) \cup f(e)^c$. For any $x \in X$, $h(e)(x) = f(e)(x) \vee (f(e)^c)(x) = f(e)(x) \vee f(e)(x)' = 1$. Then $h(e) = \widetilde{1}$. This show that $h_A = X_A$. That is, $f_A \,\widetilde{\cup}\, (f_A)^c = X_A$.
(3) This is similar to the proof of (2).
(4) Put

$$(f_A \,\widetilde{\cup}\, g_A)^c = h_A, \qquad (f_A)^c \,\widetilde{\cap}\, (g_A)^c = l_A.$$

For any $e \in A$, $h(e) = (f(e) \cup g(e))^c$, $l(e) = f(e)^c \cap g(e)^c$. And for any $x \in X$, $h(e)(x) = (f(e) \cup g(e))^c(x) = ((f(e) \cup g(e))(x))' = (f(e)(x) \vee g(e)(x))'$. By Proposition 2.5, $h(e)(x) = (f(e)(x))' \wedge (g(e)(x))'$. And $l(e)(x) = (f(e)^c \cap g(e)^c)(x) = f(e)^c(x) \wedge g(e)^c(x) = (f(e)(x))' \wedge (g(e)(x))'$. Then $h(e)(x) = l(e)(x)$. Thus $h(e) = l(e)$. This show that $h_A = l_A$. That is,

$$(f_A \,\widetilde{\cup}\, g_A)^c = (f_A)^c \,\widetilde{\cap}\, (g_A)^c.$$

(5) This is similar to the proof of (4). □

Definition 3.14. Let f_E be an L-fuzzy soft set over X.

(1) f_E is called full, if $\bigcup_{e\in E} f(e) = \widetilde{1}$.
(2) f_E is called keeping intersection, if for any $e_1, e_2 \in E$, there exists $e_3 \in E$ such that $f(e_1) \cap f(e_2) = f(e_3)$.
(3) f_E is called keeping union, if for any $e_1, e_2 \in E$, there exists $e_3 \in E$ such that $f(e_1) \cup f(e_2) = f(e_3)$.
(4) f_E is called topological, if $\{f(e) : e \in E\}$ is an L-fuzzy topology on X.

Obviously, every topological L-fuzzy soft set is full, keeping intersection and keeping union, and f_E is keeping intersection (resp. keeping union) if and only if for any $E' \subseteq E$, there exists $e' \in E$ such that $\bigcap_{e\in E'} f(e) = f(e')$ (resp. $\bigcup_{e\in E'} f(e) = f(e')$).

Example 3.15. Let L be a complete Boolean lattice depicted in Fig. 2, $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2, e_3\}$. Let f_E be an L-fuzzy soft set over X defined as follows

$$f(e_1) = \left\{\frac{x_1}{0}, \frac{x_2}{1}, \frac{x_3}{0}, \frac{x_4}{1}\right\}, \qquad f(e_2) = \widetilde{0}, \qquad f(e_3) = \left\{\frac{x_1}{1}, \frac{x_2}{0}, \frac{x_3}{1}, \frac{x_4}{0}\right\}.$$

We have $f(e_1) \cap f(e_2) = f(e_1) \cap f(e_3) = f(e_2) \cap f(e_3) = f(e_2), f(e_1) \cup f(e_3) = \widetilde{1} \neq f(e)$ $(\forall e \in E)$. Thus f_E is full and keeping intersection. But f_E is not keeping union.

Example 3.16. Let L be a complete Boolean lattice depicted in Fig. 2, $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2, e_3\}$. Let f_E be an L-fuzzy soft set over X defined as follows

$$f(e_1) = \left\{\frac{x_1}{a}, \frac{x_2}{1}, \frac{x_3}{c}, \frac{x_4}{a}\right\}, \qquad f(e_2) = \left\{\frac{x_1}{e}, \frac{x_2}{b}, \frac{x_3}{f}, \frac{x_4}{e}\right\}, \qquad f(e_3) = \left\{\frac{x_1}{e}, \frac{x_2}{b}, \frac{x_3}{f}, \frac{x_4}{0}\right\}.$$

Then f_E is keeping intersection and keeping union. But f_E is not full.

Example 3.17. Let L be a complete Boolean lattice depicted in Fig. 2, $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2, e_3\}$. Let f_E be an L-fuzzy soft set over X defined as follows

$$f(e_1) = \left\{\frac{x_1}{0}, \frac{x_2}{1}, \frac{x_3}{0}, \frac{x_4}{1}\right\}, \qquad f(e_2) = \left\{\frac{x_1}{1}, \frac{x_2}{0}, \frac{x_3}{1}, \frac{x_4}{0}\right\}, \qquad f(e_3) = \widetilde{1}.$$

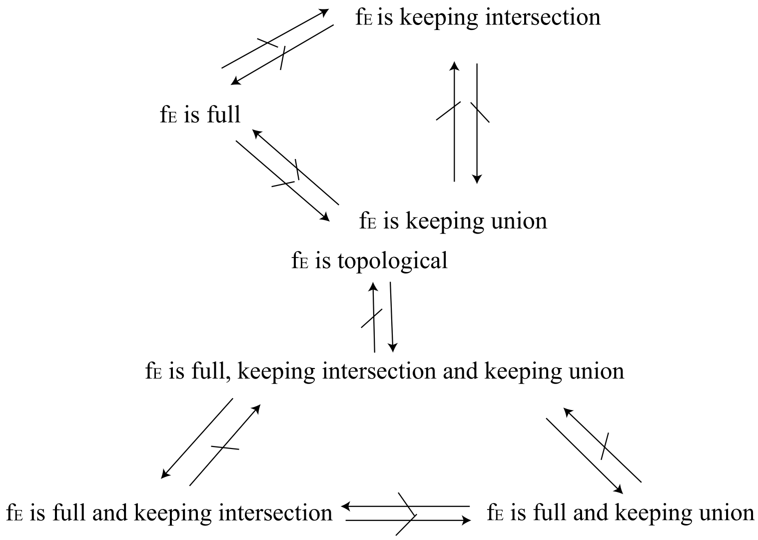
Then f_E is full and keeping union. But f_E is not keeping intersection.

Example 3.18. Let L be a complete Boolean lattice depicted in Fig. 2, $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2, e_3\}$. Let f_E be an L-fuzzy soft set over X defined as follows

$$f(e_1) = \left\{\frac{x_1}{1}, \frac{x_2}{c}, \frac{x_3}{a}, \frac{x_4}{f}\right\}, \qquad f(e_2) = \left\{\frac{x_1}{e}, \frac{x_2}{f}, \frac{x_3}{e}, \frac{x_4}{0}\right\}, \qquad f(e_3) = \widetilde{1}.$$

Then f_E is full, keeping intersection and keeping union. But f_E is not topological.

From Examples 3.15–3.18, we have the following relationships:



4. The lattice structure of L-fuzzy soft sets

In this section we investigate the lattice structure of L-fuzzy soft sets.
We denote

$S(X, E) = \{f_A : A \subseteq E$ and f_A is an L-fuzzy soft set over $X\}$.

$S_1(X, E) = \{f_E : f_E$ is an L-fuzzy soft set over $X\}$.

Obviously,

$S_1(X, E) \subseteq S(X, E)$.

Theorem 4.1. *For any $f_A, g_B \in S(X, E)$, define*

$$f_A \le g_B \Leftrightarrow f_A \mathrel{\widetilde{\subseteq}} g_B, \qquad f_A \vee g_B = f_A \mathbin{\widetilde{\cup}} g_B, \qquad f_A \wedge g_B = f_A \mathbin{\widetilde{\cap}} g_B.$$

Then $S(X, E)$ is a distributive lattice with $1_{\sum}$ and $0_{\sum}$.

Proof. Denote $\sum = S(X, E)$. It is easily proved that

$$0_{\sum} = \emptyset_{\emptyset} \quad \text{and} \quad 1_{\sum} = X_E.$$

By Proposition 3.12, $S(X, E)$ is a distributive lattice with $1_{\sum}$ and $0_{\sum}$. □

Theorem 4.2. *For any $f_E, g_E \in S_1(X, E)$, define*

$$f_E \le g_E \Leftrightarrow f_E \mathrel{\widetilde{\subseteq}} g_E, \qquad f_E \vee g_E = f_E \mathbin{\widetilde{\cup}} g_E, \qquad f_E \wedge g_E = f_E \mathbin{\widetilde{\cap}} g_E.$$

Then $S_1(X, E)$ is a Boolean lattice.

Proof. Denote $\sum_1 = S(X, E)$. It is easily proved that $S_1(X, E)$ is a distributive lattice with $0_{\sum_1} = \emptyset_E$ and $1_{\sum_1} = X_E$.

Let $f_E \in \sum_1$. Put $h_E = f_E \vee {f^c}_E$. Since $h_E = f_E \mathbin{\widetilde{\cup}} {f^c}_E$, then for any $e \in E$,

$$h(e) = f(e) \cup f^c(e) = f(e) \cup f(e)^c.$$

Thus for any $x \in X$,

$$h(e)(x) = f(e)(x) \vee f(e)^c(x) = f(e)(x) \vee (f(e)(x))' = 1.$$

This implies for any $e \in E$, $h(e) = \widetilde{1}$. So $h_E = X_E = 1_{\sum_1}$. This show that $f_E \vee {f^c}_E = 1_{\sum_1}$.

Similarly, we can prove $f_E \wedge {f^c}_E = 0_{\sum_1}$. Hence $(f_E)' = {f^c}_E$.

Therefore, $S_1(X, E)$ is a Boolean lattice. □

Corollary 4.3. *$S_1(X, E)$ is a sublattice of $S(X, E)$.*

5. *L*-fuzzy soft rough approximations and *L*-fuzzy soft rough sets

Let f_E be an L-fuzzy soft set over X, $x \in X$, $e \in E$ and $A \in L^X$. Denote

$\underline{A}_x = \{\lambda \in L : x_\lambda \in L^X \text{ and } x_\lambda \in f(e) \subseteq A \text{ for some } e \in E\}$,

$\overline{A}_x = \{\lambda \in L : x_\lambda \in L^X \text{ and } x_\lambda \in f(e) \text{ and } f(e) \cap A \neq \widetilde{0} \text{ for some } e \in E\}$;

$\underline{A}^e_x = \{\lambda \in L : x_\lambda \in L^X \text{ and } x_\lambda \in f(e) \subseteq A\}$,

$\overline{A}^e_x = \{\lambda \in L : x_\lambda \in L^X,\ x_\lambda \in f(e) \text{ and } f(e) \cap A \neq \widetilde{0}\}$;

$\underline{A}^e(x) = \vee \underline{A}^e_x, \qquad \overline{A}^e(x) = \vee \overline{A}^e_x.$

We firstly consider a pair of L-fuzzy soft rough approximations.

Definition 5.1. Let f_E be an L-fuzzy soft set over X. Then the pair $P = (X, f_E)$ is called an L-fuzzy soft approximation space. Based on the L-fuzzy soft approximation space P, we define the following two operations: for $A \in L^X$, $x \in X$,

$$\underline{\mathrm{apr}}_P(A)(x) = \vee \underline{A}_x, \qquad \overline{\mathrm{apr}}_P(A)(x) = \vee \overline{A}_x.$$

$\underline{\mathrm{apr}}_P(A)$ and $\overline{\mathrm{apr}}_P(A)$ are called the L-fuzzy soft P-lower approximation and the L-fuzzy soft P-upper approximation of X, respectively. In general, we refer to $\underline{\mathrm{apr}}_P(A)$ and $\overline{\mathrm{apr}}_P(A)$ as L-fuzzy soft rough approximations of X with respect to P.

A is called an L-fuzzy soft P-definable set, if $\underline{\mathrm{apr}}_P(A) = \overline{\mathrm{apr}}_P(A)$; A is called an L-fuzzy soft P-rough set, if $\underline{\mathrm{apr}}_P(A) \neq \overline{\mathrm{apr}}_P(A)$.

In this paper, we denote

$\mathcal{R} = \{A \in L^X : A \text{ is an } L\text{-fuzzy soft } P\text{-rough set}\}$,

$\mathcal{D} = \{A \in L^X : A \text{ is an } L\text{-fuzzy soft } P\text{-definable set}\}$,

$\tau_f = \{A \in L^X :\ A = \underline{\mathrm{apr}}_P(A)\}$,

$\sigma_f = \{A \in L^X :\ A = \overline{\mathrm{apr}}_P(A)\}$,

where $P = (X, f_E)$ is an L-fuzzy soft approximation space.

Lemma 5.2. *Let f_E be an L-fuzzy soft set over X and let $x \in X$ and $A, B \in L^X$. Then the following properties hold.*

(1) $\underline{A}_x \subseteq \overline{A}_x$.

(2) (a) $\underline{A}_x = \bigcup_{e \in E} \underline{A}^e_x$; (b) $\overline{A}_x = \bigcup_{e \in E} \overline{A}^e_x$.

(3) (a) $A \subseteq B \Rightarrow \underline{A}_x \subseteq \underline{B}_x$; (b) $A \subseteq B \Rightarrow \overline{A}_x \subseteq \overline{B}_x$.
(4) *If f_E is keeping intersection, then* $\underline{(A \cap B)}_x = \underline{A}_x \cap \underline{B}_x$.
(5) $\overline{(A \cup B)}_x = \overline{A}_x \cup \overline{B}_x$.

Proof. The proofs of (1) and (2) are obvious.

(3) (a) If $\underline{A}_x = \emptyset$, obviously, $\underline{A}_x \subseteq \underline{B}_x$. If $\underline{A}_x \neq \emptyset$, assume $\lambda \in \underline{A}_x$, we have $x_\lambda \in L^X$ and $x_\lambda \in f(e) \subseteq A$ for some $e \in E$. Since $f(e) \subseteq A$ and $A \subseteq B$, so $f(e) \subseteq B$. This implies $x_\lambda \in f(e) \subseteq B$. Thus $\lambda \in \underline{B}_x$. Hence $\underline{A}_x \subseteq \underline{B}_x$.

(b) We suppose $\overline{A}_x \neq \emptyset$ and $\lambda \in \overline{A}_x$. Then $x_\lambda \in L^X$, $x_\lambda \in f(e)$ and $f(e) \cap A \neq \widetilde{0}$ for some $e \in E$. $f(e) \cap A \neq \widetilde{0}$ implies

$$f(e)(y) \wedge A(y) = (f(e) \wedge A)(y) > 0 \quad \text{for some } y \in X.$$

Since $A \subseteq B$, then $A(y) \leq B(y)$, then

$$(f(e) \cap B)(y) = f(e)(y) \wedge B(y) \geq f(e)(y) \wedge A(y) > 0.$$

Thus $f(e) \cap B \neq \widetilde{0}$. This implies $\lambda \in \overline{B}_x$. Hence $\overline{A}_x \subseteq \overline{B}_x$.

(4) By (3), $\underline{(A \cap B)}_x \subseteq \underline{A}_x \cap \underline{B}_x$.

Conversely, we suppose $\underline{A}_x \cap \underline{B}_x \neq \emptyset$ and $\lambda \in \underline{A}_x \cap \underline{B}_x$. Then there exist $e_1, e_2 \in E$ such that $x_\lambda \in f(e_1) \subseteq A$ and $x_\lambda \in f(e_2) \subseteq B$, i.e.

$$\lambda \leq f(e_1)(x) \quad \text{and} \quad \lambda \leq f(e_2)(x).$$

This implies $\lambda \leq f(e_1)(x) \wedge f(e_2)(x) = (f(e_1) \cap f(e_2))(x)$. Because f_E is keeping intersection, we have $f(e_1) \cap f(e_2) = f(e)$ for some $e \in E$. Now $\lambda \leq f(e)(x)$. So $x_\lambda \in f(e)$. Since $f(e_1) \subseteq A$ and $f(e_2) \subseteq B$, then $f(e) \subseteq A \cap B$ and $\lambda \in \underline{(A \cap B)}_x$. Then $\underline{(A \cap B)}_x \supseteq \underline{A}_x \cap \underline{B}_x$.

(5) By (3), $\overline{(A \cup B)}_x \supseteq \overline{A}_x \cup \overline{B}_x$.

Conversely, we suppose $\overline{(A \cup B)}_x \neq \emptyset$ and $\lambda \in \overline{(A \cup B)}_x$. Then there exists $e \in E$ such that $x_\lambda \in f(e)$, $f(e) \cap (A \cup B) \neq \widetilde{0}$. Now

$$f(e) \cap (A \cup B) = (f(e) \cap A) \cup (f(e) \cap B) \neq \widetilde{0}.$$

Thus $f(e) \cap A \neq \widetilde{0}$ or $f(e) \cap B \neq \widetilde{0}$. This implies $x_\lambda \in \overline{A}_x \cup \overline{B}_x$. So we have $\overline{(A \cup B)}_x \subseteq \overline{A}_x \cup \overline{B}_x$. Hence $\overline{(A \cup B)}_x = \overline{A}_x \cup \overline{B}_x$. □

Proposition 5.3. *Let f_E be an L-fuzzy soft set over X, let $P = (X, f_E)$ be an L-fuzzy soft approximation space and $A, B \in L^X$. Then the following properties hold:*

(1) $\underline{\mathrm{apr}}_P(A) = \bigcup_{e \in E} \underline{A}^e = \bigcup\{f(e) : e \in E \textit{ and } f(e) \subseteq A\}$.
(2) $\overline{\mathrm{apr}}_P(A) = \bigcup_{e \in E} \overline{A}^e = \bigcup\{f(e) : e \in E \textit{ and } f(e) \cap A \neq \widetilde{0}\}$.
(3) $\underline{\mathrm{apr}}_P(A) \subseteq \overline{\mathrm{apr}}_P(A)$; $\underline{\mathrm{apr}}_P(A) \subseteq A$.
(4) $\underline{\mathrm{apr}}_P(\widetilde{0}) = \overline{\mathrm{apr}}_P(\widetilde{0}) = \widetilde{0}$; $\underline{\mathrm{apr}}_P(\widetilde{1}) = \overline{\mathrm{apr}}_P(\widetilde{1})$.
(5) $A \subseteq B \Rightarrow \underline{\mathrm{apr}}_P(A) \subseteq \underline{\mathrm{apr}}_P(B)$; $A \subseteq B \Rightarrow \overline{\mathrm{apr}}_P(A) \subseteq \overline{\mathrm{apr}}_P(B)$.
(6) $\overline{\mathrm{apr}}_P(A \cup B) = \overline{\mathrm{apr}}_P(A) \cup \overline{\mathrm{apr}}_P(B)$.
(7) $\underline{\mathrm{apr}}_P(\underline{\mathrm{apr}}_P(A)) = \underline{\mathrm{apr}}_P(A)$; $\underline{\mathrm{apr}}_P(\overline{\mathrm{apr}}_P(A)) = \overline{\mathrm{apr}}_P(A)$.

Proof. (1) We first prove that $\underline{\mathrm{apr}}_P(A) = \bigcup_{e \in E} \underline{A}^e$. It suffices to show that

$$\underline{\mathrm{apr}}_P(A)(x) = \vee \{\underline{A}^e(x) : e \in E\} \quad \text{for each } x \in X.$$

If $\underline{A}_x = \emptyset$, then $\underline{\mathrm{apr}}_P(A)(x) = \vee \underline{A}_x = 0$. By Lemma 5.2, $\underline{A}_x = \bigcup_{e \in E} \underline{A}^e_x$. This implies $\underline{A}^e_x = \emptyset$ for $e \in E$. Then $\underline{A}^e(x) = \vee \underline{A}^e_x = 0$ for each $e \in E$. Thus

$$\underline{\mathrm{apr}}_P(A)(x) = \vee \{\underline{A}^e(x) : e \in E\}.$$

If $\underline{A}_x \neq \emptyset$, then $\underline{\mathrm{apr}}_P(A)(x) = \vee \underline{A}_x$. Put

$$E_1 = \{e \in E : \underline{A}^e_x \neq \emptyset\}.$$

By Lemma 5.2, $\underline{A}_x = \bigcup_{e \in E} \underline{A}^e_x$. Then $E_1 \neq \emptyset$. We can suppose $E_1 \neq E$. Since $\underline{A}_x = \bigcup_{e \in E_1} \underline{A}^e_x$, then $\vee \underline{A}_x = \vee \{\vee \underline{A}^e_x : e \in E_1\}$. Note that E is a finite set and $\underline{A}^e(x) = \vee \underline{A}^e_x = 0$ for each $e \in E - E_1$. Then

$$\vee \{\underline{A}^e(x) : e \in E\} = \vee \{\underline{A}^e(x) : e \in E_1\} = \vee \{\vee \underline{A}^e_x : e \in E_1\} = \vee \underline{A}_x.$$

Thus

$$\underline{\mathrm{apr}}_P(A)(x) = \vee \{\underline{A}^e(x) : e \in E\}.$$

Secondly, we prove that $\underline{\mathrm{apr}}_P(A) = \bigcup\{f(e) : e \in E \text{ and } f(e) \subseteq A\}$.

It suffices to show that

$$\underline{\mathrm{apr}}_P(A)(x) = \vee\{f(e)(x) : e \in E_1\} \quad \text{for each } x \in X,$$

where $E_1 = \{e \in E : f(e) \subseteq A\}$.

Case 1. If $A = \widetilde{0}$, then $\underline{A}_x = \emptyset$. So $\underline{\mathrm{apr}}_P(A)(x) = \vee\emptyset = 0$. Since $\{f(e) : e \in E_1\} = \emptyset$ or $\{\widetilde{0}\}$, then $\vee\{f(e)(x) : e \in E_1\} = \vee\emptyset$ or $\vee\{f(e)(x) : e \in E'\}$, where $f(e) = \widetilde{0}$ for any $e \in E'$ and $E' \subseteq E_1$. So $\vee\{f(e)(x) : e \in E_1\} = 0$. Thus $\underline{\mathrm{apr}}_P(A)(x) = \vee\{f(e)(x) : e \in E_1\}$.

Case 2. If $A \neq \widetilde{0}$ and $E_1 = \emptyset$, then $\underline{A}_x = \emptyset$ and $\{f(e)(x) : e \in E_1\} = \emptyset$. Thus $\underline{\mathrm{apr}}_P(A)(x) = 0 = \vee\{f(e)(x) : e \in E_1\}$.

Case 3. If $A \neq \widetilde{0}, E_1 \neq \emptyset$ and $\underline{A}_x = \emptyset$, then $\underline{\mathrm{apr}}_P(A)(x) = 0$. We claim that $f(e)(x) = 0$ for any $e \in E_1$. Otherwise, $f(e)(x) \neq 0$ for some $e \in E_1$, pick $\lambda = f(e)(x)$, then $\lambda \in \underline{A}_x$. This implies $\underline{A}_x \neq \emptyset$, a contradiction. So $\vee\{f(e)(x) : e \in E_1\} = 0$. Thus $\underline{\mathrm{apr}}_P(A)(x) = \vee\{f(e)(x) : e \in E_1\}$.

Case 4. If $A \neq \widetilde{0}$, $E_1 \neq \emptyset$ and $\underline{A}_x \neq \emptyset$, then for every $\lambda \in \underline{A}_x$, there exists $e_\lambda \in E$ such that $x_\lambda \in f(e_\lambda) \subseteq A$. Note that $e_\lambda \in E_1$ and $\lambda \leq f(e_\lambda)(x)$. Then $\lambda \leq \vee\{f(e)(x) : e \in E_1\}$. Thus

$$\underline{\mathrm{apr}}_P(A)(x) = \vee\underline{A}_x \leq \vee\{f(e)(x) : e \in E_1\}.$$

Conversely, for each $y \in \{f(e)(x) : e \in E_1\}$, there exists $e_y \in E_1$ such that $y = f(e_y)(x)$.

If $f(e_y)(x) = 0$, then $y \leq \underline{\mathrm{apr}}_P(A)(x)$.

If $f(e_y)(x) \neq 0$, put $\lambda = f(e_y)(x)$, then $x_\lambda \in f(e_y) \subseteq A$. So $\lambda \in \underline{A}_x$. This implies $y = \lambda \leq \underline{\mathrm{apr}}_P(A)(x)$.

Thus

$$\underline{\mathrm{apr}}_P(A)(x) \geq \vee\{f(e)(x) : e \in E_1\}.$$

(2) This is similar to the proof of (1).

(3) By Lemma 5.2, we can easily prove that $\underline{\mathrm{apr}}_P(A) \subseteq \overline{\mathrm{apr}}_P(A)$.

Let $x \in X$. If $\underline{A}_x = \emptyset$, then $\underline{\mathrm{apr}}_P(A)(x) = 0 \leq A(x)$. If $\underline{A}_x \neq \emptyset$, for each $\lambda \in \underline{A}_x$, there exists $e_\lambda \in E$ such that $x_\lambda \in f(e_\lambda) \subseteq A$. So $\lambda \leq f(e_\lambda)(x) \leq A(x)$. This implies $\underline{\mathrm{apr}}_P(A)(x) = \vee\underline{A}_x \leq A(x)$.

(4) The proofs are obvious.

(5) Suppose $A \subseteq B$, $x \in X$ and $\underline{A}_x \neq \emptyset$. For each $\lambda \in \underline{A}_x$, $\lambda \in \underline{B}_x$ by Lemma 5.2. Then $\lambda \leq \vee\underline{B}_x = \underline{\mathrm{apr}}_P(B)(x)$. This implies $\underline{\mathrm{apr}}_P(A)(x) = \vee\underline{A}_x \leq \underline{\mathrm{apr}}_P(B)(x)$. Thus $\underline{\mathrm{apr}}_P(A) \subseteq \underline{\mathrm{apr}}_P(B)$.

Similarly, we can prove that $A \subseteq B \Rightarrow \overline{\mathrm{apr}}_P(A) \subseteq \overline{\mathrm{apr}}_P(B)$.

(6) By (5), $\overline{\mathrm{apr}}_P(A \cup B) \supseteq \overline{\mathrm{apr}}_P(A) \vee \overline{\mathrm{apr}}_P(B)$.

Conversely. Let $x \in X$. We can suppose $\overline{(A \cup B)}_x \neq \emptyset$. For each $\lambda \in \overline{(A \cup B)}_x$, we have $\lambda \in \overline{A}_x \cup \overline{B}_x$ by Lemma 5.2. Then $\lambda \in \overline{A}_x$ or $\lambda \in \overline{B}_x$. Thus

$$\lambda \leq \vee\overline{A}_x = \overline{\mathrm{apr}}_P(A)(x) \quad \text{or} \quad \lambda \leq \vee\overline{B}_x = \overline{\mathrm{apr}}_P(B)(x).$$

We have $\lambda \leq \overline{\mathrm{apr}}_P(A)(x) \vee \overline{\mathrm{apr}}_P(B)(x) = (\overline{\mathrm{apr}}_P(A) \cup \overline{\mathrm{apr}}_P(B))(x)$. This implies $\overline{\mathrm{apr}}_P(A \cup B)(x) = \vee\overline{(A \cup B)}_x \leq (\overline{\mathrm{apr}}_P(A) \cup \overline{\mathrm{apr}}_P(B))(x)$. Thus $\overline{\mathrm{apr}}_P(A \cup B) \subseteq \overline{\mathrm{apr}}_P(A) \cup \overline{\mathrm{apr}}_P(B)$.

(7) We first prove that $\underline{\mathrm{apr}}_P(\underline{\mathrm{apr}}_P(A)) = \underline{\mathrm{apr}}_P(A)$. By (3), it suffices to show that $\underline{\mathrm{apr}}_P(B)(x) \geq B(x)$ for each $x \in X$, where $B = \underline{\mathrm{apr}}_P(A)$.

Suppose $\underline{A}_x \neq \emptyset$, then for each $\lambda \in \underline{A}_x$, there exists $e_\lambda \in E$ such that $x_\lambda \in L^X$ and $x_\lambda \in f(e_\lambda) \subseteq A$.

By (1), $B = \bigcup\{f(e) : e \in E \text{ and } f(e) \subseteq A\}$. Then $f(e_\lambda) \subseteq B$. This implies $\lambda \in \underline{B}_x$. Thus $\lambda \leq \vee\underline{B}_x = \underline{\mathrm{apr}}_P(B)(x)$. Hence $B(x) = \vee\underline{A}_x \leq \underline{\mathrm{apr}}_P(B)(x)$.

By (2), we can similarly prove that $\underline{\mathrm{apr}}_P(\overline{\mathrm{apr}}_P(A)) = \overline{\mathrm{apr}}_P(A)$. □

Example 5.4. Let L be a complete Boolean lattice depicted in Fig. 2, $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2\}$. Let f_E be an L-fuzzy soft set over X defined as follows

$$f(e_1) = \left\{\frac{x_1}{0}, \frac{x_2}{f}, \frac{x_3}{b}, \frac{x_4}{0}\right\}, \qquad f(e_2) = \left\{\frac{x_1}{f}, \frac{x_2}{c}, \frac{x_3}{f}, \frac{x_4}{a}\right\}.$$

(1) Let

$$A = \left\{\frac{x_1}{c}, \frac{x_2}{0}, \frac{x_3}{0}, \frac{x_4}{a}\right\}.$$

We have $f(e_1) \cap A = \widetilde{0},\ f(e_2) \cap A \neq \widetilde{0}$. Then $\overline{A}_{x_1} = \overline{A}^{e_2}_{x_1} = \{\lambda \in L : \lambda \leq f(e_2)(x_1)\} = \{f\}$. So $\overline{\mathrm{apr}}_P(A)(x_1) = \vee\overline{A}_{x_1} = f < c = A(x_1)$. Thus $A \nsubseteq \overline{\mathrm{apr}}_P(A)$.

(2) Since $f(e_1) \subseteq \widetilde{1},\ f(e_2) \subseteq \widetilde{1}$, then

$$\begin{aligned}\underline{\widetilde{1}}_{x_2} &= \{\lambda \in L : \lambda \leq f(e_1)(x_2)\} \cup \{\lambda \in L : \lambda \leq f(e_2)(x_2)\}\\ &= \{f\} \cup \{c, d, f\} = \{c, d, f\}.\end{aligned}$$

This implies $\underline{\mathrm{apr}}_P(\widetilde{1})(x_2) = \vee \underline{\widetilde{1}}_{x_2} = c < 1 = \widetilde{1}(x_2)$.
Thus

$$\underline{\mathrm{apr}}_P(\widetilde{1}) \neq \widetilde{1}.$$

(3) Let $B = \left\{\frac{x_1}{e}, \frac{x_2}{c}, \frac{x_3}{f}, \frac{x_4}{a}\right\}$. Then $A \cup B = \left\{\frac{x_1}{1}, \frac{x_2}{c}, \frac{x_3}{f}, \frac{x_4}{a}\right\}$. We have $f(e_1) \not\subseteq A,\ f(e_2) \not\subseteq A;\ f(e_1) \not\subseteq B,\ f(e_2) \not\subseteq B;\ f(e_1) \not\subseteq A \cup B,\ f(e_2) \subseteq A \cup B$.
$\underline{A}_{x_2} = \emptyset$ implies $\underline{\mathrm{apr}}_P(A)(x_2) = 0$.
$\underline{B}_{x_2} = \emptyset$ implies $\underline{\mathrm{apr}}_P(B)(x_2) = 0$.

$$\underline{(A \cup B)}_{x_2} = \underline{(A \cup B)}^{e_2}_{x_2} = \{c, d, f\}$$

implies $\underline{\mathrm{apr}}_P(A \cup B)(x_2) = \vee \underline{(A \cup B)}_{x_2} = c$.
Note that $(\underline{\mathrm{apr}}_P(A) \cup \underline{\mathrm{apr}}_P(B))(x_2) = 0 \neq \underline{\mathrm{apr}}_P(A \cup B)(x_2) = c$.
Thus

$$\underline{\mathrm{apr}}_P(A) \cup \underline{\mathrm{apr}}_P(B) \neq \underline{\mathrm{apr}}_P(A \cup B).$$

(4) Let

$$C = \left\{\frac{x_1}{f}, \frac{x_2}{0}, \frac{x_3}{b}, \frac{x_4}{a}\right\}; \qquad D = \left\{\frac{x_1}{f}, \frac{x_2}{a}, \frac{x_3}{0}, \frac{x_4}{f}\right\}.$$

Then

$$C \cap D = \left\{\frac{x_1}{f}, \frac{x_2}{0}, \frac{x_3}{0}, \frac{x_4}{f}\right\}.$$

We have

$$\begin{aligned}&f(e_1) \cap C \neq \widetilde{0}, \quad f(e_2) \cap C \neq \widetilde{0}; \quad f(e_1) \cap D \neq \widetilde{0}, \quad f(e_2) \cap D \neq \widetilde{0};\\ &f(e_1) \cap (C \cap D) = \widetilde{0}, \quad f(e_2) \cap (C \cap D) \neq \widetilde{0}.\end{aligned}$$

$\overline{C}_{x_3} = \overline{D}_{x_3} = \{b, d, e, f\}$ implies $\overline{\mathrm{apr}}_P(C)(x_3) = \overline{\mathrm{apr}}_P(D)(x_3) = 1$.
$\overline{(C \cap D)}_{x_3} = \overline{(C \cap D)}^{e_2}_{x_3} = \{c, d, f\}$ implies $\overline{\mathrm{apr}}_P(C \cap D)(x_3) = c$.
Note that

$$(\overline{\mathrm{apr}}_P(C) \cap \overline{\mathrm{apr}}_P(D))(x_3) = 1 \neq \overline{\mathrm{apr}}_P(C \cap D)(x_3) = c.$$

Thus

$$\overline{\mathrm{apr}}_P(C) \cap \overline{\mathrm{apr}}_P(D) \neq \overline{\mathrm{apr}}_P(C \cap D).$$

Example 5.5. Let L be a complete Boolean lattice depicted in Fig. 2, $X = \{x_1, x_2, x_3, x_4\}$ and $E = \{e_1, e_2\}$. Let f_E be an L-fuzzy soft set over X defined as follows

$$f(e_1) = \left\{\frac{x_1}{e}, \frac{x_2}{c}, \frac{x_3}{e}, \frac{x_4}{c}\right\}, \qquad f(e_2) = \left\{\frac{x_1}{b}, \frac{x_2}{f}, \frac{x_3}{e}, \frac{x_4}{1}\right\}.$$

Let

$$A = \left\{\frac{x_1}{e}, \frac{x_2}{c}, \frac{x_3}{b}, \frac{x_4}{c}\right\}, \qquad B = \left\{\frac{x_1}{b}, \frac{x_2}{f}, \frac{x_3}{b}, \frac{x_4}{1}\right\}.$$

Then

$$A \cap B = \left\{\frac{x_1}{e}, \frac{x_2}{f}, \frac{x_3}{b}, \frac{x_4}{c}\right\}.$$

We have $f(e_1) \subseteq A,\ f(e_2) \not\subseteq A;\ f(e_1) \not\subseteq B,\ f(e_2) \subseteq B;\ f(e_1) \not\subseteq A \cap B,\ f(e_2) \not\subseteq A \cap B$.
$\underline{A}_{x_2} = \underline{A}^{e_1}_{x_2} = \{\lambda \in L : \lambda \leq f(e_1)(x_2)\} = \{c, d, f\}$ implies

$$\underline{\mathrm{apr}}_P(A)(x_2) = \vee \underline{A}_{x_2} = c.$$

$\underline{B}_{x_2} = \underline{B}^{e_2}_{x_2} = \{\lambda \in L : \lambda \leq f(e_2)(x_2)\} = \{f\}$ implies

$$\underline{\mathrm{apr}}_P(B)(x_2) = \vee \underline{B}_{x_2} = f.$$

$\underline{(A \cap B)}_{x_2} = \emptyset$ implies $\underline{\mathrm{apr}}_P(A \cap B)(x_2) = 0$.

Note that

$$(\underline{\mathrm{apr}}_P(A) \cap \underline{\mathrm{apr}}_P(B))(x_2) = f \neq \underline{\mathrm{apr}}_P(A \cap B)(x_2) = 0.$$

Thus $\underline{\mathrm{apr}}_P(A) \cap \underline{\mathrm{apr}}_P(B) \neq \underline{\mathrm{apr}}_P(A \cap B)$.

Proposition 5.6. *Let f_E be an L-fuzzy soft set over X and let $P = (X, f_E)$ be an L-fuzzy soft approximation space. Then the following properties hold:*

(1) *If f_E is keeping intersection, then*

$$\underline{\mathrm{apr}}_P(A \cap B) = \underline{\mathrm{apr}}_P(A) \cap \underline{\mathrm{apr}}_P(B) \quad \textit{for any } \mathcal{A}, B \in L^X.$$

(2) *If f_E is full and keeping union, then*

(a) $\overline{\mathrm{apr}}_P(A) = \widetilde{1}$ *for each* $A \in L^X \setminus \{\widetilde{0_L}\}$;

(b) $\underline{\mathrm{apr}}_P(A) \subseteq A \subseteq \overline{\mathrm{apr}}_P(A)$ *for any* $A \in L^X$;

(c) $\underline{\mathrm{apr}}_P(\widetilde{1}) = \overline{\mathrm{apr}}_P(\widetilde{1}) = \widetilde{1}$.

Proof. (1) By Proposition 5.3, $\underline{\mathrm{apr}}_P(A \cap B) \subseteq \underline{\mathrm{apr}}_P(A) \cap \underline{\mathrm{apr}}_P(B)$. We only need to prove the converse. By the distributivity of Boolean lattices, we obtain that for each $x \in X$,

$$\begin{aligned}
(\underline{\mathrm{apr}}_P(A) \cap \underline{\mathrm{apr}}_P(B))(x) &= \underline{\mathrm{apr}}_P(A)(x) \wedge \underline{\mathrm{apr}}_P(B)(x) \\
&= \left(\bigvee\{f(e)(x) : e \in E \text{ and } f(e) \subseteq A\}\right) \wedge \left(\bigvee\{f(e')(x) : e' \in E \text{ and } f(e') \subseteq B\}\right) \\
&= \bigvee\{f(e)(x) \wedge f(e')(x) : e, e' \in E,\ f(e) \subseteq A \text{ and } f(e') \subseteq B\} \\
&= \bigvee\{(f(e) \cap f(e'))(x) : e, e' \in E,\ f(e) \subseteq A \text{ and } f(e') \subseteq B\}.
\end{aligned}$$

Since $P = (X, f_E)$ is keeping intersection, then for any $e, e' \in E$, there exists $e'' \in E$ such that $f(e) \cap f(e') = f(e'') \subseteq A \cap B$. Thus

$$\underline{\mathrm{apr}}_P(A \cap B)(x) \geq (\underline{\mathrm{apr}}_P(A) \cap \underline{\mathrm{apr}}_P(B))(x).$$

Hence

$$\underline{\mathrm{apr}}_P(A \cap B) \geq \underline{\mathrm{apr}}_P(A) \cap \underline{\mathrm{apr}}_P(B).$$

(2) (a) Since f_E is full and keeping union, then $\widetilde{1} = \bigcup_{e \in E} f(e) = f(e^*)$ for some $e^* \in E$. For each $A \in L^X \setminus \{\widetilde{0_L}\}$ and each $x \in X$, pick $\lambda = 1$, then $x_\lambda \in f(e^*)$ and $f(e^*) \cap A = A \neq \widetilde{0}$. This implies $\lambda \in \overline{A}_x$. So $\overline{\mathrm{apr}}_P(A)(x) = 1$. Thus $\overline{\mathrm{apr}}_P(A) = \widetilde{1}$.

(2) (b) This holds by Proposition 5.3 and (2)(a).

(2) (c) This holds by (2)(a). □

Theorem 5.7. *Let f_E be an L-fuzzy soft set over X and let $P = (X, f_E)$ be an L-fuzzy soft approximation space. Then for each $A \in L^X$, A is an L-fuzzy soft P-definable set if and only if $\overline{\mathrm{apr}}_P(A) \subseteq A$.*

Proof. Note that if A is L-fuzzy soft P-definable, then $\underline{\mathrm{apr}}_P(A) = \overline{\mathrm{apr}}_P(A)$. By Proposition 5.3, $\underline{\mathrm{apr}}_P(A) \subseteq A$. Thus $\overline{\mathrm{apr}}_P(A) \subseteq A$.

Conversely, suppose $\overline{\mathrm{apr}}_P(A) \subseteq A$. To prove that A is L-fuzzy soft P-definable, we only need to show that

$$\overline{\mathrm{apr}}_P(A)(x) = \underline{\mathrm{apr}}_P(A)(x) \quad \text{for each } x \in X.$$

Put $B = \overline{\mathrm{apr}}_P(A)$, we claim that $\overline{A}_x \subseteq \underline{B}_x$. In fact, we can suppose $\overline{A}_x \neq \emptyset$. For each $\lambda \in \overline{A}_x, x_\lambda \in L^X$ and $x_\lambda \in f(e_\lambda)$, $f(e_\lambda) \cap A \neq \widetilde{0}$ for some $e_\lambda \in E$. By Proposition 5.3,

$$B = \bigcup\{f(e) : e \in E \text{ and } f(e) \cap A \neq \widetilde{0}\}.$$

Then $f(e_\lambda) \subseteq B$. This implies $\lambda \in \underline{B}_x$. Thus $\overline{A}_x \subseteq \underline{B}_x$.

Note that $B \subseteq A$, by Lemma 5.2, $\underline{B}_x \subseteq \underline{A}_x$. Since $\underline{A}_x \subseteq \overline{A}_x \subseteq \underline{B}_x$, then $\overline{A}_x = \underline{A}_x$. Hence

$$\overline{\mathrm{apr}}_P(A)(x) = \vee \overline{A}_x = \vee \underline{A}_x = \underline{\mathrm{apr}}_P(A)(x). \quad \square$$

Corollary 5.8. *Let f_E be an L-fuzzy soft set over X and let $P = (X, f_E)$ be an L-fuzzy soft approximation space. Then for each $A \in L^X$, A is an L-fuzzy soft P-rough set if and only if $\overline{\mathrm{apr}}_P(A) \nsubseteq A$.*

Below, we give the structure of L-fuzzy soft rough sets.
For $\mathcal{P}, \mathcal{Q} \subseteq L^X$, denote

$$\mathcal{P} \setminus \mathcal{Q} = \{A \in L^X : A \in \mathcal{P} \text{ and } A \notin \mathcal{Q}\}.$$

Theorem 5.9. *Let f_E be an L-fuzzy soft set over X and let $P = (X, f_E)$ be a soft approximation space. Then*
(1)

$$\mathcal{R} \cup \mathcal{D} = L^X, \qquad \mathcal{R} \cap \mathcal{D} = \emptyset \quad \text{and} \quad \sigma_f \subseteq \mathcal{D}.$$

(2) *If f_E is full and keeping union, then*

$$\mathcal{R} = L^X \setminus \{\widetilde{0}, \widetilde{1}\} \quad \text{and} \quad \mathcal{D} = \sigma_f = \{\widetilde{0}, \widetilde{1}\} \subseteq \tau_f.$$

Proof. This holds by Propositions 5.3, 5.6 and Theorem 5.7. □

Remark 5.10. Theorem 5.9 gives the structure of L-fuzzy soft rough sets.

6. The relationship between *L*-fuzzy soft sets and *L*-fuzzy topologies

In this section we investigate the relationship between L-fuzzy soft sets and L-fuzzy topologies.

Theorem 6.1. *Let f_E be an L-fuzzy soft set over X and $P = (X, f_E)$ be an L-fuzzy soft approximation space. Then*

(1) *τ_f is a generalized L-fuzzy topology on X.*
(2) *If f_E is full, keeping intersection and keeping union, then τ_f is an L-fuzzy topology on X.*
(3) *If f_E is full and keeping union, then $\sigma_f = \{\widetilde{0}, \widetilde{1}\}$ is an indiscreet L-fuzzy topology on X.*

Proof. (1) By Proposition 5.3, $\widetilde{0} \in \tau$.
Let $\{A_\alpha : \alpha \in \Gamma\} \subseteq \tau$. Put $A = \cup\{A_\alpha : \alpha \in \Gamma\}$. Since $A_\alpha \subseteq A$ for each $\alpha \in \Gamma$, we have $A_\alpha = \underline{\mathrm{apr}}_P(A_\alpha) \subseteq \underline{\mathrm{apr}}_P(A)$ by Proposition 5.3. So $A = \vee\{A_\alpha : \alpha \in \Gamma\} \subseteq \underline{\mathrm{apr}}_P(A)$. By Proposition 5.3, $\underline{\mathrm{apr}}_P(A) \subseteq A$. Thus $\underline{\mathrm{apr}}_P(A) = A$. This implies $\vee\{A_\alpha : \alpha \in \Gamma\} \in \tau$. Hence τ is a generalized L-fuzzy topology on X.
(2) By Propositions 5.3 and 5.6, we have $\widetilde{1} \in \tau$ and $A \cap B \in \tau$ whenever $A, B \in \tau$. By (1), τ_f is a generalized L-fuzzy topology on X. Thus τ is an L-fuzzy topology on X.
(3) This holds by Theorem 5.9. □

Definition 6.2. Let f_E be a full, keeping intersection and keeping union L-fuzzy soft set over X and $P = (X, f_E)$ be an L-fuzzy soft approximation space. Then τ_f is called the L-fuzzy topology induced by f_E on X.

The following Theorem 6.3 gives the topological structure on L-fuzzy soft sets (i.e., the structure of the L-fuzzy topology induced by an L-fuzzy soft set).

Theorem 6.3. *Let τ_f be the L-fuzzy topology induced by a full, keeping intersection and keeping union f_E on X. Then*

(1)

$$\{\overline{\mathrm{apr}}_P(A) : A \in L^X\} \subseteq \tau_f = \{\underline{\mathrm{apr}}_P(A) : A \in L^X\}.$$

(2)

$$\tau_f \supseteq \{f(e) : e \in E\}.$$

(3) *If f_E is topological, then*

$$\tau_f = \{f(e) : e \in E\}.$$

(4) *$\underline{\mathrm{apr}}_P$ is an interior operator of τ_f.*

Proof. (1) By Proposition 5.3, we have $\{\overline{\mathrm{apr}}_P(A) : A \in L^X\} \subseteq \tau_f$.
Obviously, $\tau_f \subseteq \{\underline{\mathrm{apr}}_P(A) : A \in L^X\}$.
Let $B \in \{\underline{\mathrm{apr}}_P(A) : A \in L^X\}$. Then $B = \underline{\mathrm{apr}}_P(A)$ for some $A \in L^X$. By Proposition 5.3, $\underline{\mathrm{apr}}_P(\underline{\mathrm{apr}}_P(A)) = \underline{\mathrm{apr}}_P(A)$. This implies $B \in \tau_f$. Thus $\tau_f \supseteq \{\underline{\mathrm{apr}}_P(A) : A \in L^X\}$.
(2) For each $e \in E$, by Proposition 5.3, $\underline{\mathrm{apr}}_P(f(e)) = f(e)$. So we have $f(e) \in \tau_f$. Hence $\{f(e) : e \in E\} \subseteq \tau_f$.
(3) By (2), $\tau_f \supseteq \{f(e) : e \in E\}$.

Let $A \in \tau_f$. If $A = \widetilde{0}$, by f_E is topological, then $A \in \{f(e) : e \in E\}$.

If $A \neq \widetilde{0}$, then $A = \underline{\mathrm{apr}}_P(A)$. Put $E_1 = \{e \in E : f(e) \subseteq A\}$, by Proposition 5.3, $A = \cup\{f(e) : e \in E_1\}$. We claim that $E_1 \neq \emptyset$. Otherwise, $E_1 = \emptyset$. Then for each $x \in X$, $A(x) = \vee\{f(e)(x) : e \in E_1\} = \vee\emptyset = 0$. So $A = \widetilde{0}$, a contradiction. Since f_E is keeping union, then there exists $e' \in E$ such that $\cup\{f(e) : e \in E_1\} = f(e')$. That is, $A = f(e')$.

Thus $\tau_f \subseteq \{f(e) : e \in E\}$.

Hence

$$\tau_f = \{f(e) : e \in E\}.$$

(4) It suffices to show that $\underline{\mathrm{apr}}_P(A) = \mathrm{int}(A)$ for each $A \in L^X$.

By (1), $\underline{\mathrm{apr}}_P(A) \in \tau_f$. By Proposition 5.3, $\underline{\mathrm{apr}}_P(A) \subseteq A$. Thus $\underline{\mathrm{apr}}_P(A) \subseteq \mathrm{int}(A)$.

Conversely. For each $B \in \tau_f$ with $B \subseteq A$, we have $B = \underline{\mathrm{apr}}_P(B) \subseteq \underline{\mathrm{apr}}_P(A)$ by Proposition 5.3. Thus $\mathrm{int}(A) = \bigcup\{B : B \in \tau_f \text{ and } B \subseteq A\} \subseteq \underline{\mathrm{apr}}_P(A)$. Hence $\underline{\mathrm{apr}}_P(A) = \mathrm{int}(A)$. □

We recall that an L-fuzzy topology is finite if this L-fuzzy topology has only finite elements.

Definition 6.4. Let τ be a finite L-fuzzy topology on X. Denote $\tau = \{U_e : e \in E\}$, where E is the set of indexes. Define a mapping $f_\tau : E \to L^X$ by $f_\tau(e) = U_e$ for each $e \in E$. Then the L-fuzzy soft set $(f_\tau)_E$ over X is called the L-fuzzy soft set induced by τ on X.

Definition 6.5. Let (X, μ) be an L-fuzzy topological space. If there exists a full, keeping intersection and keeping union L-fuzzy soft set f_E over X such that $\tau_f = \mu$, then (X, μ) is called an L-fuzzy soft approximating space.

The following Proposition 6.6 can easily be proved.

Proposition 6.6. (1) *Let τ be a finite L-fuzzy topology on X and let $(f_\tau)_E$ be the L-fuzzy soft set induced by τ on X. Then $(f_\tau)_E$ is topological.*

(2) *Let τ_1 and τ_2 be two finite L-fuzzy topologies on X and let $(f_{\tau_1})_{E_1}$ and $(f_{\tau_1})_{E_2}$ be two L-fuzzy soft sets induced respectively by τ_1 and τ_2 on X. If $\tau_1 \subseteq \tau_2$, then*

$$(f_{\tau_1})_{E_1} \,\widetilde{\subset}\, (f_{\tau_2})_{E_2}.$$

Theorem 6.7. *Let τ be a finite L-fuzzy topology on X. Then $\tau = \tau_{f_\tau}$.*

Proof. Put $\tau = \{U_e : e \in E\}$, then $f_\tau : E \to L^X$ is a mapping, where $f_\tau(e) = U_e$ for each $e \in E$. By Proposition 6.6, $(f_\tau)_E$ is full, keeping intersection and keeping union. By Theorem 6.3, $\tau_{f_\tau} = \{f_\tau(e) : e \in E\}$. Hence $\tau_{f_\tau} = \tau$. □

Theorem 6.8. *Let τ be a finite L-fuzzy topology on X. Then there exists a full, keeping intersection and keeping union L-fuzzy soft set f_E over X such that*

$$\underline{\mathrm{apr}}_P(A) = \mathrm{int}(A) \quad \text{for each } A \in L^X,$$

where $P = (X, f_E)$ is a soft approximation space.

Proof. Put $\tau = \{U_e : e \in E\}$, where E is the set of indexes. Define a mapping $f : E \to L^X$ by $f(e) = U_e$ for each $e \in E$. By Proposition 6.6, f_E is full, keeping intersection and keeping union. Assume $A \in L^X$,

$$\underline{\mathrm{apr}}_P(A) = \cup\{f(e) : f(e) \subseteq A \text{ and } e \in E\} = \cup\{U_e : U_e \subseteq A \text{ and } e \in E\}.$$

Thus $\underline{\mathrm{apr}}_P(A) = \mathrm{int}(A)$. □

Corollary 6.9. *Every finite L-fuzzy topological space is an L-fuzzy soft approximating space.*

Theorem 6.10. *Let f_E be a full, keeping intersection and keeping union L-fuzzy soft set over X, let τ_f be the L-fuzzy topology induced by f_E on X and let $(f_{\tau_f})_{E'}$ be the L-fuzzy soft set induced by τ_f on X. Then*

(1)

$$f_E \,\widetilde{\subset}\, (f_{\tau_f})_{E'}.$$

(2) *If f_E is topological, then*

$$f_E = (f_{\tau_f})_{E'}.$$

Proof. (1) By Theorem 6.3, $\tau_f \supseteq \{f(e) : e \in E\}$. Denote

$$\tau_f = \{U_e : e \in E'\}, \quad \text{where } E \subseteq E' \quad \text{and} \quad U_e = f(e) \quad \text{for each } e \in E.$$

Thus f_{τ_f} is a mapping given by

$$f_{\tau_f} : E' \to L^X, \quad \text{where } f_{\tau_f}(e) = U_e \text{ for each } e \in E'.$$

Hence

$$f_E \mathrel{\widetilde{\subset}} (f_{\tau_f})_{E'}.$$

(2) Since f_E is topological, then by Theorem 6.3, $E = E'$. Hence

$$f_E = (f_{\tau_f})_{E'}. \quad \square$$

7. Some correspondence relationships associated with L-fuzzy soft sets

In this section we obtain some correspondence relationships associated with L-fuzzy soft sets.

Definition 7.1 ([20]). Let X be a finite set of objects and let E be a finite set of attributes. The pair (X, E, V, g) is called an information system, if $V = \bigcup_{e\in E} V_e$ is the value domain of E, and g is an information function specifying the attributes value for each object and is given by $g : X \times E \to V$, where $V_e = \{g(x, e) : x \in X\}$ is the value domain of the attribute e.

Definition 7.2. Let (X, E, V, g) be an information system.

(1) (X, E, V, g) is called a 2-valued information system, if $V = \{0, 1\}$.
(2) (X, E, V, g) is called a $[0, 1]$-valued information system, if $V \subseteq [0, 1]$.
(3) (X, E, V, g) is called an L-valued information system, if $V \subseteq L$.

Obviously, every 2-valued information system is a $[0, 1]$-valued information system and every $[0, 1]$-valued information system is an L-valued information system.

Proposition 7.3 ([21]). *Every soft set may be considered a 2-valued information system.*

Proposition 7.4. *Every L-fuzzy soft set may be considered an L-valued information system.*

Proof. Let $f_{E'}$ be an L-fuzzy soft set over X' and let (X, E, L, g) be an L-valued information system. Obviously, the universe X' in $f_{E'}$ may be considered the set X of objects in (X, E, L, g), and the set E' of parameters may be considered the set E of attributes in (X, E, L, g). The information function g is defined by

$$g(x, e) = f(e)(x)$$

for each $(e, x) \in E \times X$.

That is, $V = \bigcup_{e\in E} V_e$ is the value domain of E, where $V_e = \{f(e)(x) : x \in X\}$ is the value domain of the attribute e. Obviously, $V \subseteq L$.

Therefore, $f_{E'}$ may be considered an L-valued information system (X, E, L, g). $\square$

Definition 7.5. Let $S = f_E$ be an L-fuzzy soft set over X. Define a function $g_S : X \times E \to V$ by

$$g_S(x, e) = f(e)(x)$$

for each $(e, x) \in E \times X$. Then (X, E, L, g_S) is called an L-valued information system induced by S. We denote it by IS_S.

Definition 7.6. Let $\mathrm{IS} = (X, E, L, g)$ be an L-valued information system. Define a mapping $f_{\mathrm{IS}} : E \to I^X$ by

$$f_{\mathrm{IS}}(e)(x) = g(x, e)$$

$(x \in X)$ for each $e \in E$, then $(f_{\mathrm{IS}})_E = (f_{\mathrm{IS}}, E)$ is called an L-fuzzy soft set over E induced by IS. We denote it by S_{IS}.

Lemma 7.7. *Let $S = f_E$ be an L-fuzzy soft set over E, let $\mathrm{IS}_S = (X, E, L, g_S)$ be an L-valued information system induced by S over X and let S_{IS_S} be an L-fuzzy soft set over X induced by IS_S. Then $S = S_{\mathrm{IS}_S}$.*

Proof. Obviously, $S_{\mathrm{IS}_S} = (f_{\mathrm{IS}_S}, E)$.

For each $e \in E$,

$$f_{\mathrm{IS}_S}(e)(x) = g_S(x, e) \quad (\forall x \in X).$$

Since $g_S(x, e) = f(e)(x)$, then for each $e \in E$, $f_{\mathrm{IS}_S}(e)(x) = f(e)(x)$ $(\forall x \in X)$. So $f_{\mathrm{IS}_S}(e) = f(e)$ for each $e \in E$. This implies $f_{\mathrm{IS}_S} = f$. Hence $S = S_{\mathrm{IS}_S}$. $\square$

Lemma 7.8. *Let $\mathrm{IS} = (X, E, L, g)$ be an L-valued information system, let S_{IS} be an L-fuzzy soft set over X induced by IS and let $\mathrm{IS}_{S_{\mathrm{IS}}} = (X, E, L, g_{S_{\mathrm{IS}}})$ be an L-valued information system induced by S_{IS}. Then $\mathrm{IS} = \mathrm{IS}_{S_{\mathrm{IS}}}$.*

Proof. Obviously, $\mathrm{IS}_{S_{\mathrm{IS}}} = (X, E, L, g_{S_{\mathrm{IS}}})$.

For any $(x, e) \in X \times E$,

$$g_{S_{\mathrm{IS}}}(x, e) = f_{\mathrm{IS}}(e)(x).$$

Since for any $x \in X$, $e \in E$, $f_{\mathrm{IS}}(e)(x) = g(x, e)$, then $g_{S_{\mathrm{IS}}}(x, e) = g(x, e)$. This implies $g_{S_{\mathrm{IS}}} = g$. Hence

$$\mathrm{IS} = \mathrm{IS}_{S_{\mathrm{IS}}}. \quad \square$$

Theorem 7.9. *Let $\Sigma = \{S : S = f_E$ is an L-fuzzy soft set over $X\}$ and $\Gamma = \{\mathrm{IS} : \mathrm{IS} = (X, E, L, g)$ is an L-valued information system$\}$. Then there exists a one-to-one correspondence between Σ and Γ.*

Proof. Two mapping $F : \Sigma \to \Gamma$ and $G : \Gamma \to \Sigma$ are defined as follows:

$$F(S) = \mathrm{IS}_S \quad \text{for any } S \in \Sigma,$$
$$G(\mathrm{IS}) = S_{\mathrm{IS}} \quad \text{for any } \mathrm{IS} \in \Gamma.$$

By Lemma 7.7,

$$G \circ F = i_\Sigma,$$

where $G \circ F$ is the composition of F and G, and i_Σ is the identity mapping on Σ.

By Lemma 7.8,

$$F \circ G = i_\Gamma,$$

where $G \circ F$ is the composition of G and F, and i_Γ is the identity mapping on Γ.

Hence F and G are both a one-to-one correspondence. This proves that there exists a one-to-one correspondence between Σ and Γ. $\square$

We recall that an L-fuzzy relation R from E to X is a mapping $R : E \times X \to L$.

Definition 7.10. Let f_E be an L-fuzzy soft set over X. Define an L-fuzzy relation ρ_f from E to X by $\rho_f(e, x) = f(e)(x)$ for each $(e, x) \in E \times X$. Then ρ_f is called the L-fuzzy relation from E to X induced by f_E.

Definition 7.11. Let ρ be an L-fuzzy relation from E to X. Define a mapping $f_\rho : E \to L^X$ by $f_\rho(e)(x) = \rho(e, x)$ $(x \in X)$ for each $e \in E$. Then $(f_\rho)_E$ is called the L-fuzzy soft set induced by ρ.

Lemma 7.12. *Let ρ be an L-fuzzy relation from E to X, let $(f_\rho)_E$ be the L-fuzzy soft set induced by ρ and let ρ_{f_ρ} be the L-fuzzy relation from E to X induced by $(f_\rho)_E$. Then $\rho = \rho_{f_\rho}$.*

Proof. For each $(e, x) \in E \times X$, $\rho_{f_\rho}(e, x) = f_\rho(e)(x)$. Note that $f_\rho(e)(x) = \rho(e, x)$. Then $\rho_{f_\rho}(e, x) = \rho(e, x)$. Thus $\rho = \rho_{f_\rho}$. $\square$

Lemma 7.13. *Let f_E be an L-fuzzy soft set over X, let ρ_f be the L-fuzzy relation from E to X induced by f_E and let $(f_{\rho_f})_E$ be the L-fuzzy soft set induced by ρ_f. Then*

$$f_E = (f_{\rho_f})_E.$$

Proof. Since f_{ρ_f} is a mapping given by $f_{\rho_f} : E \to L^X$, where $f_{\rho_f}(e)(x) = \rho_f(e, x)$ $(x \in X)$ for each $e \in E$, then $f_{\rho_f}(e)(x) = f(e)(x)$ for each $x \in X$. Then $f_{\rho_f}(e) = f_\rho(e)$ for each $e \in E$. Thus $f_E = (f_{\rho_f})_E$. $\square$

Similar to the proof of Theorem 7.9, we can prove the following Theorem 7.14 by Lemmas 7.12 and 7.13.

Theorem 7.14. *Let $\Sigma = \{f_E : f_E$ is an L-fuzzy soft set over $X\}$ and $\Gamma = \{\rho : \rho$ is an L-fuzzy relation from E to $X\}$. Then there exists a one-to-one correspondence between Σ and Γ.*

8. Conclusions

In this paper, we introduced the concept of L-fuzzy soft sets as a generalization of fuzzy soft sets, and obtained the lattice and topological structure of L-fuzzy soft sets. We considered a pair of L-fuzzy soft rough approximations and gave their properties. We revealed the fact that every finite L-fuzzy topological space is an L-fuzzy soft approximating space. We introduced the concept of L-fuzzy soft rough sets and studied their structures. In addition, we proved that there exists a one-to-one correspondence between the set of all L-fuzzy soft sets and the set of all L-valued information systems. We also proved that there exists a one-to-one correspondence between the set of all L-fuzzy soft sets and the set of all L-fuzzy relations from the set of parameters to the initial universe. We may mention that these correspondences are of theoretical significance for the study of soft set theory and information systems. We will investigate these problems in future papers.

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